BI-LIPSCHITZ EMBEDDING OF THE GENERALIZED GRUSHIN PLANE INTO EUCLIDEAN SPACES

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Abstract. We show that, for all $\alpha \geq 0$, the generalized Grushin plane $\mathbb{G}_\alpha$ is bi-Lipschitz homeomorphic to a 2-dimensional quasiplane in the Euclidean space $\mathbb{R}^{[\alpha]+2}$, where $[\alpha]$ is the integer part of α. The target dimension is sharp. This generalizes a recent result of Wu [22].

1. Introduction

The classical Grushin plane $\mathbb{G}$ is defined as the space $\mathbb{R}^2$ equipped with the sub-Riemannian (Carnot-Carathéodory) metric $d_{\mathbb{G}}$ generated by the vector fields

$$X_1 = \partial_{x_1} \quad \text{and} \quad X_2 = x_1\partial_{x_2}.$$

This means more precisely that the distance between points $p, q \in \mathbb{G}$ is

$$d_{\mathbb{G}}(p,q) = \inf_{\gamma} \int_0^1 \sqrt{x_1'(t)^2 + \frac{x_2'(t)^2}{x_1(t)^2}}dt,$$

where the infimum is taken over all paths $\gamma = (x_1(t), x_2(t)) \colon [0,1] \to \mathbb{G}$, with $\gamma(0) = p$ and $\gamma(1) = q$, that are absolutely continuous in the Euclidean metric. The Grushin plane is one of the simplest examples of a sub-Riemannian manifold, as well as a basic example of the *almost Riemannian* manifolds studied by Agrachev, Boscain, Charlot, Ghezzi, and Sigalotti [2], [3]. For additional background on the Grushin plane and sub-Riemannian spaces in general, see Bellaïche [6].

Recently, Seo [18] proved a general characterization of spaces admitting a bi-Lipschitz embedding into some Euclidean space $\mathbb{R}^n$, from which it follows that $\mathbb{G}$ admits such an embedding. In contrast, the Heisenberg group can not be embedded bi-Lipschitz in any Euclidean space [17]. While Seo's result does not give the optimal target dimension, Wu [22] constructed an explicit bi-Lipschitz embedding of $\mathbb{G}$ into $\mathbb{R}^3$, where the dimension 3 is the smallest possible.

In the present article, Wu's result is extended to the generalized Grushin plane $\mathbb{G}_\alpha$, $\alpha \geq 0$, studied first by Franchi and Lanconelli [11]. Similarly to $d_{\mathbb{G}}$, the metric $d_{\mathbb{G}_\alpha}$ is generated by the vector fields

$$X_1 = \partial_{x_1} \quad \text{and} \quad X_2 = |x_1|^\alpha \partial_{x_2}.$$

For integer values of α, $|x_1|^\alpha$ can be replaced by x_1^α and the space $\mathbb{G}_\alpha$ is a sub-Riemannian manifold of step $\alpha + 1$. For noninteger values of α,

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this space is technically not sub-Riemannian, but this distinction does not matter for the purposes of this paper. Meyerson [15] and Ackermann [1] have shown that $\mathbb{G}_\alpha$ is quasisymmetric to the Euclidean space $\mathbb{R}^2$ for any $\alpha \geq 0$. Moreover, it can be deduced by Seo's theorem [18, Theorem 4.4] that $\mathbb{G}_\alpha$ is bi-Lipschitz embeddable into some Euclidean space when $\alpha > 0$, though without identifying the smallest target dimension.

In this paper, we construct for each $\alpha \geq 0$ a bi-Lipschitz embedding of $\mathbb{G}_\alpha$ into $\mathbb{R}^{[\alpha]+2}$ where $[\alpha]$ is the greatest integer that is less or equal to α. A point of interest in both Wu's and our construction is that the image of $\mathbb{G}_\alpha$ is a quasiplane in $\mathbb{R}^{[\alpha]+2}$.

Theorem 1.1. *For all integers $N \geq 0$ and $n \geq 1$, there exists $L > 1$ depending only on N, n such that for any $\alpha \in [N, N + \frac{n-1}{n}]$, there exists an L-bi-Lipschitz homeomorphism of $\mathbb{G}_\alpha$ onto a 2-dimensional quasiplane $\mathcal{P}_\alpha$ in $\mathbb{R}^{N+2}$.*

A *k-dimensional quasiplane* $\mathcal{P}$ in $\mathbb{R}^n$, with $k < n$, is the image of a k-dimensional hyperplane in $\mathbb{R}^n$ under a quasiconformal self-map of $\mathbb{R}^n$. Complete characterizations of these spaces in terms of their geometric structure exist only for $n = 2$, $k = 1$ by Ahlfors [4]. While such intrinsic characterizations have been elusive for $n \geq 3$, several intriguing examples of quasiplanes and quasispheres have been constructed [8, 10, 13, 14, 16, 21, 22].

A couple of remarks are in order. The target dimension $N + 2 = [\alpha] + 2$ in Theorem 1.1 is minimal. Indeed, by (5.1), the *singular line* $\{x_1 = 0\}$ of $\mathbb{G}_\alpha$ is bi-Lipschitz homeomorphic to the "snowflaked" space $(\mathbb{R}, |\cdot|^{1/(1+\alpha)})$ which, by a well-known theorem of Assouad [5, Proposition 4.12], embeds bi-Lipschitz into $\mathbb{R}^{[\alpha]+2}$ with the target dimension $[\alpha]+2$ being the smallest possible when $\alpha > 0$. It is noteworthy that, for $\alpha > 0$, $\mathbb{G}_\alpha$ embeds in the same Euclidean space that its singular line embeds in.

The same result of Assouad also justifies the dependence of the constant L on n. For if there was a uniform L such that $\mathbb{G}_\alpha$ was L-bi-Lipschitz embeddable in $\mathbb{R}^{N+2}$ for all $\alpha \in [N, N+1)$, then by a simple Arzelà-Ascoli limiting argument (see Lemma 5.4), it would follow that $\mathbb{G}_{N+1}$, thus the singular line in $\mathbb{G}_{N+1}$, is also embeddable in $\mathbb{R}^{N+2}$ which is false.

The following corollary is an immediate consequence of Theorem 1.1.

Corollary 1.2. *If $\alpha \in [0, 1)$ then $\mathbb{G}_\alpha$ is bi-Lipschitz homeomorphic to $\mathbb{R}^2$.*

Therefore, $\mathbb{G}_\alpha$ is bi-Lipschitz homeomorphic to $\mathbb{G}_\beta$ whenever $\alpha, \beta \in [0, 1)$. In contrast, if $\alpha \geq 1$ then $\mathbb{G}_\alpha$ has Hausdorff dimension $\alpha + 1$, and is bi-Lipschitz homeomorphic to $\mathbb{G}_\beta$ only when $\alpha = \beta$. Combined with the Beurling-Ahlfors quasiconformal extension [7], Corollary 1.2 yields the following result.

Corollary 1.3. *If $\alpha \in [0, 1)$, then any bi-Lipschitz embedding of the singular line of $\mathbb{G}_\alpha$ into $\mathbb{R}^2$ extends to a bi-Lipschitz homeomorphism of $\mathbb{G}_\alpha$ onto $\mathbb{R}^2$.*

An alternative proof of Corollary 1.2 along with new results on questions of quasisymmetric parametrizability and bi-Lipschitz embeddability of high-dimensional Grushin spaces can be found in a recent paper of Wu [23].

1.1. **Outline of the proof of Theorem 1.1.** The proof of Theorem 1.1 comprises two parts. In Section 5.1 we show Theorem 1.1 for rationals $\alpha \geq 0$ and in Section 5.2 we use an Arzelà-Ascoli limiting argument to prove Theorem 1.1 for all real values $\alpha \geq 0$. The proof of Corollary 1.3 is also given in Section 5.

Much of the proof of Theorem 1.1 for rational $\alpha \geq 0$ follows the method of Wu in [22]. The crux of the proof is the construction, for each rational $\alpha \in [N, N + \frac{n-1}{n}]$, of a quasisymmetric mapping $F_\alpha : \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$, such that in each ball $B(x, \frac{1}{2}\operatorname{dist}(x, \{0\} \times \mathbb{R}))$, F_α is the product of $\operatorname{dist}(x, \{0\} \times \mathbb{R})^{-\frac{1}{1+\alpha}}$ and a λ-bi-Lipschitz mapping with λ depending only on N, n. Such a mapping is $\frac{1}{1+\alpha}$-*snowflaking* on $\{0\} \times \mathbb{R} \subset \mathbb{R}^{N+1} \times \mathbb{R}$ (i.e. $|F_\alpha(x) - F_\alpha(y)| \simeq |x-y|^{\frac{1}{1+\alpha}}$ for all $x, y \in \{0\} \times \mathbb{R}$) and maps the 2-dimensional plane $\mathbb{R} \times \{0\} \times \mathbb{R}$ onto a quasiplane $\mathcal{P}_\alpha$. Composed with a quasisymmetric homeomorphism of $\mathbb{G}_\alpha$ onto $\mathbb{R}^2$, we obtain a bi-Lipschitz homeomorphism f_α of $\mathbb{G}_\alpha$ into $\mathcal{P}_\alpha$.

The quasisymmetric mappings F_α are constructed in Section 4 by iterating a finite number of bi-Lipschitz mappings Θ which are defined in Section 3 as in [22]. However, a straightforward generalization of Wu's method, without additional care, would give no control on the local bi-Lipschitz constant λ (thus on the bi-Lipschitz constant of f_α), and the proof of Theorem 1.1 for irrational values of α would not be possible. To overcome this issue, we construct in Section 3 two sets of bi-Lipschitz mappings Θ_z, corresponding to $z = 0$ and $z = N + \frac{n-1}{n}$, and then periodically alternate between these when constructing the quasisymmetric mapping F_α.

Our inability to define F_α for irrational $\alpha > 0$ is the reason for considering the irrational case separately; see Remark 4.6.

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2. Preliminaries

A homeomorphism $f \colon D \to D'$ between two domains in $\mathbb{R}^n$ is called K-*quasiconformal* if it is orientation-preserving, belongs to $W^{1,n}_{\mathrm{loc}}(D)$, and satisfies the distortion inequality

$$|Df(x)|^n \leq K J_f(x) \quad \text{a. e. } x \in D,$$

where Df is the formal differential matrix and J_f is the Jacobian.

An embedding f of a metric space (X, d_X) into a metric space (Y, d_Y) is said to be η-*quasisymmetric* if there exists a homeomorphism $\eta \colon [0, \infty) \to [0, \infty)$ such that for all $x, a, b \in X$ and $t > 0$ with $d_X(x, a) \leq t d_X(x, b)$,

$$d_Y(f(x), f(a)) \leq \eta(t) d_Y(f(x), f(b)).$$

A quasisymmetric mapping between two domains in $\mathbb{R}^n$ is quasiconformal. On the other hand, a quasiconformal mapping defined on a domain $D \subset \mathbb{R}^n$ is quasisymmetric on each compact set $E \subset D$. In $\mathbb{R}^n$ the two notions coincide: if $f : \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiconformal then it is η-quasisymmetric for some η depending only on K, n. For a systematic treatment of quasiconformal mappings see [19].

A mapping $f\colon X \to Y$ between metric spaces is *L-bi-Lipschitz* if there exists a constant $L \geq 1$ such that $L^{-1}d_X(x,y) \leq d_Y(f(x),f(y)) \leq Ld_X(x,y)$ for all $x,y \in X$.

In the following, we write $u \lesssim v$ (resp. $u \simeq v$) when the ratio u/v is bounded above (resp. bounded above and below) by positive constants. These constants may vary, but are described in each occurrence.

3. Basic geometric constructions

This section extends the construction by Wu [22] to higher-dimensional targets; the notational conventions follow those of Wu as much as possible. Our goal is to build certain annular tubes and bi-Lipschitz maps between these tubes which are used in Section 4 to define quasiconformal homeomorphisms of $\mathbb{R}^{N+2}$. These constructions are based on examples of Bonk and Heinonen [9] and Assouad [5].

3.1. **Definitions and notation.** An *N-cube* $\mathcal{C}$ is the product $\Delta_1 \times \cdots \times \Delta_N$ of bounded closed intervals $\Delta_i \subset \mathbb{R}$ of equal length. A *j-face* of $\mathcal{C}$ is a product $\Delta_1' \times \cdots \times \Delta_N'$ where, for j indices, $\Delta_i' = \Delta_i$ and for the other $N-j$ indices Δ_i' is an endpoint of Δ_i. The 0-faces of a cube $\mathcal{C}$ are its *vertices*.

For an N-cube $\mathcal{C}$ and integer $0 \leq k \leq N$, we define a *k-flag* of $\mathcal{C}$ to be a sequence $\{\mathcal{C}^j\}_{j=0}^k$ where $\mathcal{C}^j$ is a j-face of $\mathcal{C}$ and $\mathcal{C}^{j-1} \subset \mathcal{C}^j$ for all $1 \leq j \leq k$. Observe that for N-cubes $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ and $(N-2)$-flags $\{\mathcal{C}^j\}$ and $\{\widetilde{\mathcal{C}}^j\}$, there exists a unique orientation-preserving similarity $\psi : \mathbb{R}^N \to \mathbb{R}^N$ such that $\psi(\mathcal{C}) = \widetilde{\mathcal{C}}$ and $\psi(\mathcal{C}^j) = \widetilde{\mathcal{C}}^j$ for each $0 \leq j \leq N-2$.

For a point $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and a number $r > 0$, define the cube

$$\mathcal{C}^N(x,r) = [x_1 - r/2, x_1 + r/2] \times \cdots \times [x_N - r/2, x_N + r/2]$$

and denote $\mathfrak{C}^N = \mathcal{C}^N(0,1)$ where 0 here denotes the origin in $\mathbb{R}^N$.

Slightly abusing the notation, we define for two numbers $0 < r < R < \infty$ the *cubic annulus*

$$\mathcal{A}^N(r,R) = \overline{(R\mathfrak{C}^N) \setminus (r\mathfrak{C}^N)} = [-R/2, R/2]^N \setminus (-r/2, r/2)^N.$$

Here and for the rest, for $X \subset \mathbb{R}^N$ and $c > 0$, we write $cX = \{cx : x \in X\}$.

Finally, for a polygonal arc $\ell \subset \mathbb{R}^N$ and some $\epsilon > 0$, define the *cubic thickening* of ℓ

$$\mathcal{T}^N(\ell,\epsilon) = \overline{\bigcup \mathcal{C}^N(x,\epsilon)}$$

where the union is taken over all $x \in \ell$ such that their distances from the endpoints of ℓ are at least $\epsilon/2$.

For the rest of Section 3 we fix integers $N \geq 0$, $n \geq 1$ and set

$$p = p_{N,n} = N + \frac{n-1}{n} \text{ and } M = M_{N,n} = 9^{n(N+2)}.$$

The dependence of quantities and sets on N, n is omitted whenever possible.

3.2. **Blocks.** Let $I \subset \mathfrak{C}^{N+1} \times [0,1]$ be the straight-line path from $(0,\dots,0)$ to $(0,\dots,0,1)$ and $L \subset \mathfrak{C}^{N+1} \times [0,1]$ be the straight-line path from $(0,\dots,0)$ to $(0,\dots,0,\frac{1}{2})$ concatenated with the straight-line path from $(0,\dots,0,\frac{1}{2})$ to $(\frac{1}{2},0,\dots,0,\frac{1}{2})$.

We define three types of blocks that are used throughout the paper:

(1) the *I-block* $Q_I = \mathcal{T}^{N+2}(I, \frac{M-2}{M}) = (\frac{M-2}{M}\mathfrak{C}^{N+1}) \times [0,1]$;
(2) the *L-block* $Q_L = \mathcal{T}^{N+2}(L, \frac{M-2}{M})$

$$= (\frac{M-2}{M}\mathfrak{C}^{N+1} \times [0, \frac{M-1}{M}]) \cup ([\frac{1}{2}, 1] \times \frac{M-2}{M}\mathfrak{C}^{N+1})$$

(3) the *regular block* $\mathsf{Q} = \mathfrak{C}^{N+1} \times [0,1]$.

On each of these blocks, the entrance, exit and side are defined as follows.

(1) The *entrance* of Q_I is $\mathrm{en}(Q_I) = Q_I \cap \{x_{N+2} = 0\}$,
(2) the *exit* of Q_I is $\mathrm{ex}(Q_I) = Q_I \cap \{x_{N+2} = 1\}$,
(3) the *side* of Q_I is $\mathrm{s}(Q_I) = \overline{\partial Q_I \setminus (\mathrm{en}(Q_I) \cup \mathrm{ex}(Q_I))}$.

Analogous definitions can be made for Q_L and Q with the difference that the *exit* of Q_L is $\mathrm{ex}(Q_L) = Q_L \cap \{x_1 = \frac{1}{2}\}$. These definitions are applied to images of the respective objects under similarity maps. For a similarity map h and $\ell \in \{h(I), h(L)\}$, we write Q_ℓ in place of $h(Q_I)$ or $h(Q_L)$. We call Q_ℓ the *block associated with the segment* ℓ; note that Q_ℓ naturally inherits a direction from ℓ.

3.3. **Cores.** From each block Q_I, Q_L and Q we remove a *core* from its interior, which we describe in this section.

In Section 6 we construct a simple polygonal path $J_I = J_I(N,n) \subset Q_I$ from $(0,\dots,0,0)$ to $(0,\dots,0,1)$ consisting of M^{1+p} many I- and L-segments $\ell_1,\dots,\ell_{M^{1+p}}$ of length $1/M$ labelled according to their order in J_I with the following properties.

(1) The segments ℓ_1, $\ell_{M^{1+p}}$, and $\ell_{(M^{1+p}+1)/2}$ are I-segments.
(2) For all $1 \le m < M^{1+p}$, $Q_{\ell_m} \cap Q_{\ell_{m+1}}$ is the exit of Q_{ℓ_m} and the entrance of $Q_{\ell_{m+1}}$. If $1 \le l, m \le M^{1+p}$ and $|m-l| > 1$, then $Q_{\ell_m} \cap Q_{\ell_l} = \emptyset$.
(3) $\mathrm{en}(Q_{\ell_1}) = Q_{\ell_1} \cap \partial Q_I \subset \mathrm{en}(Q_I)$ and $\mathrm{ex}(Q_{\ell_{M^{1+p}}}) = Q_{\ell_{M^{1+p}}} \cap \partial Q_I \subset \mathrm{ex}(Q_I)$. For $2 \le m \le M^{1+p}-1$, $Q_{\ell_m} \cap \partial Q_I = \emptyset$.
(4) J_I is symmetric with respect to the plane $x_{N+2} = \frac{1}{2}$.
(5) J_I is unknotted in Q_I, in the sense that there every bi-Lipschitz homeomorphism $\theta : (\partial Q_I, J_I) \to (\partial \mathsf{Q}, I)$ extends to a bi-Lipschitz homeomorphism $\Theta : Q_I \to \mathsf{Q}$.

Similarly, in Section 6 we construct a simple polygonal path $J_L = J_L(N,n) \subset Q_L$ satisfying the same properties, except that $\ell_{(M^{1+p}+1)/2}$ is an L-segment and J_L is symmetric with respect to the plane $x_1 + x_{N+2} = \frac{1}{2}$.

Given $J_I = \bigcup_{m=1}^{M^{1+p}} \ell_m$ as above, define the *core*

$$\kappa_p(Q_I) = \bigcup_{m=1}^{M^{1+p}} Q_{\ell_m} = \mathcal{T}^{N+2}(J_I, \frac{M-2}{M^2}).$$

We similarly define the core $\kappa_p(Q_L)$. The entrance, the exit and the side of $\kappa_p(Q_I), \kappa_p(Q_L)$ are canonically defined. A second set of cores $\kappa_0(Q_I), \kappa_0(Q_L)$ in Q_I, Q_L, respectively, is defined as follows. Write $I = \bigcup_{m=1}^{M} \ell_m$ with $\ell_m = \{0\} \times [m-1, m] \subset \mathbb{R}^{N+1} \times I$ and set

$$\kappa_0(Q_I) = \bigcup_{m=1}^{M} Q_{\ell_m} = \mathcal{T}^{N+2}(I, \frac{M-2}{M^2}).$$

Similarly write $L = \bigcup_{m=1}^{M} \ell'_m$ where ℓ'_m is an L-segment if $m = \frac{M+1}{2}$ and an I-segment otherwise; and each ℓ'_m has length $1/M$. Set $\kappa_0(Q_L) = \bigcup_{m=1}^{M} Q_{\ell'_m} = \mathcal{T}^{N+2}(L, \frac{M-2}{M^2})$.

To simplify the notation, in what follows we write Q_m instead of Q_{ℓ_m}.

Two types of cores are similarly defined for the regular block Q. For each $z \in \{0, p\}$ let

$$\mathsf{k}_z(\mathsf{Q}) = (M^{-1-z}\mathfrak{C}^{N+1}) \times [0,1]$$

which is composed of M^{1+z} consecutive blocks

$$\mathsf{Q}_m = M^{-1-z}\left(\mathfrak{C}^{N+1} \times [m-1, m]\right), \quad m = 1, \dots, M^{1+z}.$$

3.4. **Flag-edges.** We introduce in this section *flag-edges* and *flag-paths*, which generalize the *edges* and *edge paths* used by Wu [22, Section 2.3] to blocks of arbitrary dimensions. These play an important bookkeeping role later when defining bi-Lipschitz maps between annular tubes.

For the rest fix an $(N-1)$-flag $\mathcal{F}_0 = \{\mathcal{C}^j\}_{j=1}^{N-1}$ of $\mathfrak{C}^{N+1}$. We call the collection of faces $e_{\mathcal{F}_0} = \{(\frac{M-2}{M}\mathcal{C}^j) \times [0,1]\}_{j=0}^{N-1}$ a *flag-edge* on Q_I.

Before defining flag-edges on Q_L, we first define faces on the side $\mathrm{s}(Q_L)$ inductively. If $\mathcal{C}^0 = (x_1, \dots, x_{N+1})$, $x_j \in \{\pm\frac{M-2}{2M}\}$, is a 0-face of $\mathrm{en}(Q_L)$ then define the L-type path

$$P(\mathcal{C}^0) = (\{(x_1, \dots, x_{N+1})\} \times [0, \frac{1}{2} - x_1]) \cup ([x_1, \frac{1}{2}] \times \{(x_2, \dots, x_{N+1}, \frac{1}{2} - x_1)\}.$$

Suppose that for every j-face $\mathcal{C}^j$ of $\mathrm{en}(Q_L)$, the set $P(\mathcal{C}^j)$ has been defined. Let $\mathcal{C}^{j+1}$ be a $(j+1)$-face of $\mathrm{en}(Q_L)$ and let $\mathcal{C}^j_1, \dots, \mathcal{C}^j_{2(j+1)}$ be the j-faces of $\mathcal{C}^{j+1}$. Then define $P(\mathcal{C}^{j+1})$ to be the union of all line segments with endpoints on $\bigcup_{i=1}^{2(j+1)} P(\mathcal{C}^j_i)$ that lie entirely on $\mathrm{s}(Q_L)$. We call $P(C^{j+1})$ a $(j+2)$-face on ∂Q_L.

Let now $\mathcal{F} = \{\mathcal{C}^j\}_{j=0}^{N-1}$ be an $(N-1)$-flag of $\mathfrak{C}^{N+1}$. We call the collection $e_{\mathcal{F}} = \{P(\frac{M-2}{M}\mathcal{C}^j)\}_{j=0}^{N-1}$ a flag-edge on Q_L.

We now define *flag-paths* along the cores $\kappa_z(Q_I)$, $\kappa_z(Q_L)$ for each value $z \in \{0, p\}$. We start with the Q_L case. Rescaling an $(N-1)$-flag $\mathcal{F}$ of $\mathfrak{C}^{N+1}$, we obtain an $(N-1)$-flag $\mathcal{F}_1$ on the entrance of the first block Q_1 of $\kappa_z(Q_L)$. For a j-face $\mathcal{C}^j_1 \in \mathcal{F}_1$ define $P(\mathcal{C}^j_1) \subset \mathrm{s}(Q_1)$ as above and note that $P(\mathcal{C}^j_1)$ defines uniquely a j-face $\mathcal{C}^j_2$ on the entrance of the block Q_2. Continuing inductively we obtain j-faces $\mathcal{C}^j_m$ on $\mathrm{en}(Q_m)$ and $(j+1)$-faces $P(\mathcal{C}^j_m)$ on $\mathrm{s}(Q_m)$. Define the *flag-path* $w_{\mathcal{F}} = \{\bigcup_{m=1}^{M^{1+z}} P(\mathcal{C}^j_m)\}_{j=0}^{N-1}$. For each block Q_m in $\kappa_z(Q_L)$, $m \in \{1, \dots, M^{1+z}\}$, we call $w_{\mathcal{F}} \cap Q_m$ the *marked flag-edge of Q_m derived from the data $(Q_L, e_{\mathcal{F}})$*.

A corresponding flag-path $w_{\mathcal{F}_0}$ is defined similarly for $\kappa_z(Q_I)$. For this we use the flag $\mathcal{F}_0$ instead of an arbitrary $(N-1)$-flag $\mathcal{F}$ of $\mathfrak{C}^{N+1}$.

In addition, let $\mathsf{e} = \{\mathcal{C}^j \times [0,1] : \mathcal{C}_j \in \mathcal{F}_0\}$ be a flag-edge of Q and $\mathsf{w} = \{(M^{-1-z}\mathcal{C}^j) \times [0,1] : \mathcal{C}_j \in \mathcal{F}_0\}$ be a flag-path along $\mathsf{k}_z(\mathsf{Q})$. As before we omit the dependency on z in the notation for w.

3.5. **Annular tubes.** For each $z \in \{0, p\}$, define the *annular tubes*

$$\tau_z(Q_I) = \overline{Q_I \setminus \kappa_z(Q_I)},\ \tau_z(Q_L) = \overline{Q_L \setminus \kappa_z(Q_L)} \text{ and } \mathsf{t}_z(\mathsf{Q}) = \overline{\mathsf{Q} \setminus \mathsf{k}_z(\mathsf{Q})}.$$

For $Q \in \{Q_I, Q_L\}$, we define the *entrance* and *exit* of each $\tau_z(Q)$ as $\mathrm{en}(Q) \cap \tau_z(Q)$ and $\mathrm{ex}(Q) \cap \tau_z(Q)$, respectively. These are isometric to the cubic annulus $A = \frac{M-2}{M}\mathcal{A}^{N+1}(\frac{1}{M}, 1)$. The remaining part of $\partial\tau_z(Q_I)$ is composed of the side $\mathrm{s}(Q_I)$ of block Q_I and the side $\mathrm{s}(\kappa_z(Q_I))$ of the core $\kappa_z(Q_I)$. The boundary of $\tau_z(Q_L)$ can be similarly partitioned.

Define similarly the entrance and exit of $\mathsf{t}_z(\mathsf{Q})$. These are isometric to the cubic annulus $\mathsf{A}_z = \mathcal{A}^{N+1}(M^{-1-z}, 1)$. Note that A_z depends on z while A does not.

If σ is a similarity mapping of Q_I onto some block $\sigma(Q_I)$, we denote by $\kappa_z(\sigma(Q_I))$ the image $\sigma(\kappa_z(Q_I))$ with $z \in \{0, p\}$. The sets $\kappa_z(\sigma(Q_L))$, $\mathsf{k}_z(\sigma(\mathsf{Q}))$, $\tau_z(\sigma(Q_I))$, $\tau_z(\sigma(Q_L))$ and $\mathsf{t}_z(\sigma(\mathsf{Q}))$ are defined similarly when σ is a similarity mapping.

3.6. **Bi-Lipschitz maps between annular tubes.** For each $z \in \{0, p\}$, each $Q \in \{Q_I, Q_L\}$, and every $(N-1)$-flag $\mathcal{F}$ of $\mathfrak{C}^{N+1}$, we define in this section bi-Lipschitz homeomorphisms $\Theta_z^{\mathcal{F}} : (\mathsf{t}_z(\mathsf{Q}), \mathsf{e}, \mathsf{w}) \to (\tau_z(Q), e_{\mathcal{F}}, w_{\mathcal{F}})$ where $\mathcal{F} = \mathcal{F}_0$ if $Q = Q_I$.

The construction of these maps is performed in 4 steps. In Step 1 we define the mappings on $\mathrm{s}(\mathsf{Q})$, in Step 2 we define them on $\mathrm{s}(\mathsf{k}_z(\mathsf{Q}))$ and in Step 3 we define them on the entrance and exit of $\mathsf{t}_z(\mathsf{Q})$. Combining the first three steps we obtain bi-Lipschitz mappings $\theta_z^{\mathcal{F}} : (\partial\mathsf{t}_z, \mathsf{e}, \mathsf{w}) \to (\partial\tau_z(Q), e_{\mathcal{F}}, w_{\mathcal{F}})$. Finally, in Lemma 3.1 we extend the mappings on the whole $\mathsf{t}_z(\mathsf{Q})$.

For each $(N-1)$-flag $\mathcal{F}$ of $\mathfrak{C}^{N+1}$ let $\psi_{\mathcal{F}}$ be the unique rotation on $\mathbb{R}^{N+2}$ that maps $\mathfrak{C}^{N+1}$ onto itself and $\mathcal{F}$ onto $\mathcal{F}_0$.

Step 1. Define $\theta_z^{\mathcal{F}_0} : (\mathrm{s}(\mathsf{Q}), \mathsf{e}) \to (\mathrm{s}(Q_I), e_{\mathcal{F}_0})$ by $\theta_z^{\mathcal{F}_0}(x, t) = (\frac{M-2}{M}x, t)$, where $x \in \partial\mathfrak{C}^{N+1}$ and $t \in [0, 1]$. To define $\theta_z^{\mathcal{F}}$ onto $(\mathrm{s}(Q_L), e_{\mathcal{F}})$ first observe that $\mathrm{s}(Q_L)$ is the union of L-type 1-fibers

$$\begin{aligned} L_x =& (\{\frac{M-2}{M}(x_1, \ldots, x_{N+1})\} \times [0, 1/2 - \frac{M-2}{M}x_1]) \\ &\cup ([\frac{M-2}{M}x_1, 1/2] \times \{(0, \ldots, 0, 1/2) + \frac{M-2}{M}(x_2, \ldots, x_{N+1}, -x_1)\} \end{aligned}$$

where $x = (x_1, \ldots, x_{N+1}) \in \partial\mathfrak{C}^{N+1}$. Similarly, $\mathrm{s}(\mathsf{Q})$ is the union of 1-fibers $I_x = \{x\} \times [0, 1]$ where $x \in \partial\mathfrak{C}^{N+1}$. Define $\theta_z^{\mathcal{F}}$ on $\mathrm{s}(\mathsf{Q})$ by mapping each I_x to $L_{\psi_{\mathcal{F}}(x)}$ by arc-length parametrization. Note that for this step, the mappings $\theta_z^{\mathcal{F}}$ do not actually depend on z.

Step 2. We extend each $\theta_z^{\mathcal{F}}$ to the inner side $\mathrm{s}(\mathsf{k}_z(\mathsf{Q}))$ of $\partial\mathsf{t}_z(\mathsf{Q})$. Given a block Q_m of $\mathsf{k}_z(\mathsf{Q})$ let ζ_z^m be the similarity map in $\mathbb{R}^{N+2}$ that maps (Q, e) onto $(\mathsf{Q}_m, \mathsf{w} \cap \mathsf{Q}_m)$. Similarly, given a block Q_m in the core $\kappa_z(Q)$, let $\varepsilon(Q_m)$ be the marked flag-edge $w_{\mathcal{F}} \cap Q_m$ derived from $(Q, e_{\mathcal{F}})$ and let $\sigma_z^m : (Q(Q_m), e_{\mathcal{F}(Q_m)}) \to (Q_m, \varepsilon(Q_m))$ be a similarity map for some unique $Q(Q_m) \in \{Q_I, Q_J\}$ and $(N-1)$ flag $\mathcal{F}(Q_m)$ of $\mathfrak{C}^{N+1}$. (The dependence of σ_z^m on $\mathcal{F}$ is omitted to simplify the notation.) Define now $\theta_z^{\mathcal{F}}$ on the inner side $\mathrm{s}(\mathsf{k}_z(\mathsf{Q}))$ by taking $\theta_z^{\mathcal{F}}|\mathrm{s}(\mathsf{Q}_m) = \sigma_z^m \circ \theta_z^{\mathcal{F}(Q_m)} \circ (\zeta_z^m)^{-1}$ for all $1 \le m \le M^{1+z}$. Since the union of marked flag-edges of the Q_m is the flag-path $w_{\mathcal{F}}$, the map $\theta_z^{\mathcal{F}}$ is defined consistently on the intersection of consecutive blocks and thus is well-defined.

Step 3. For each $(N-1)$-flag $\mathcal{F}$ of $\mathfrak{C}^{N+1}$, let $\phi_z^{\mathcal{F}} : \mathsf{A}_z \to A$ with

$$\phi_z^{\mathcal{F}}(xt) = \frac{M-2}{M}\left(\frac{M-1}{M-M^{-z}}(t-1)+1\right)\psi_{\mathcal{F}}(x)$$

where $t \in [M^{-1-z}, 1]$ and $x \in \partial\mathfrak{C}^{N+1}$. Define $\theta_z^{\mathcal{F}}$ on the entrance and exit of $\mathsf{t}_z(\mathsf{Q})$ by $\phi_z^{\mathcal{F}}$ modulo an isometry chosen in such a way that the mappings $\theta_z^{\mathcal{F}} : (\partial\mathsf{t}_z(\mathsf{Q}), \mathsf{e}, \mathsf{w}) \to (\partial\tau_z(Q), e_{\mathcal{F}}, w_{\mathcal{F}})$ are homeomorphisms. Then $\theta_z^{\mathcal{F}}$ are in fact bi-Lipschitz.

The final step in the construction of the mappings $\Theta_z^{\mathcal{F}}$ is given in the next lemma.

Lemma 3.1. *Every bi-Lipschitz map $\theta_z^{\mathcal{F}}$ extends to a bi-Lipschitz map*

$$\Theta_z^{\mathcal{F}} : (\mathsf{t}_z(\mathsf{Q}), \mathsf{e}, \mathsf{w}) \to (\tau_z(Q), e_{\mathcal{F}}, w_{\mathcal{F}}).$$

We verify the lemma first for $z=0$. If $Q = Q_I$ then define $\Theta_0^{\mathcal{F}_0} : \mathsf{t}_0(\mathsf{Q}) \to \tau_0(Q_I)$ with $\Theta_0^{\mathcal{F}_0}(xt, t') = (\phi_{0,\mathcal{F}_0}(xt), t')$. If $Q = Q_L$ then note that $\tau_0(Q_I)$ is the union of 1-fibers I_x and $\tau_0(Q_L)$ is the union of 1-fibers L_x where I_x, L_x are as in Step 1 and $x \in \frac{M-2}{M}\mathcal{A}^{N+1}(M^{-1}, 1)$. Let $\theta_{\mathcal{F}} : \tau_0(Q_I) \to \tau_0(Q_L)$ be the bi-Lipschitz mapping that maps each I_x to $L_{\psi_{\mathcal{F}}^{-1}(x)}$ by arc-length parametrization. Set $\Theta_0^{\mathcal{F}} = \theta_{\mathcal{F}} \circ \Theta_0^{\mathcal{F}_0}$.

The proof of Lemma 3.1 when $z=p$ relies on the structure of the paths J_I, J_L and is deferred until Section 6.3.

4. Quasisymmetric snowflaking homeomorphisms in $\mathbb{R}^{N+2}$

The key part of the proof of Theorem 1.1 for a rational $\alpha \in [N, N+\frac{n-1}{n}]$ is the construction of a quasisymmetric mapping $F_\alpha \colon \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$ that maps Whitney squares of a 2-dimensional plane of $\mathbb{R}^{N+2}$ into sets which are bi-Lipschitz homeomorphic to the Whitney squares of $\mathbb{G}_\alpha$.

Proposition 4.1. *For all integers $N \geq 0$ and $n \geq 1$, there exists $\lambda > 1$ depending only on N, n satisfying the following. For each rational $\alpha \in [N, N+\frac{n-1}{n}]$, there exists an η-quasisymmetric map $F_\alpha : \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$ with η depending only on N, n such that,*

$$|x'|^{\frac{\alpha}{1+\alpha}} F_\alpha|_{B(x,\frac{1}{2}|x'|)} \quad \textit{is } \lambda\textit{-bi-Lipschitz} \tag{4.1}$$

for all $x = (x', x'') \in \mathbb{R}^{N+1} \times \mathbb{R}$ with $|x'| \neq 0$.

The mapping F_α is $\frac{1}{1+\alpha}$-snowflaking on $\{0\} \times \mathbb{R} \subset \mathbb{R}^{N+1} \times \mathbb{R}$.

Corollary 4.2. *Let $F_\alpha \colon \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$ be the mapping of Proposition 4.1. Then, there exists $\lambda' > 1$ depending only on N, n such that*

$$(\lambda')^{-1}|x-y|^{\frac{1}{1+\alpha}} \leq |F_\alpha(x) - F_\alpha(y)| \leq \lambda'|x-y|^{\frac{1}{1+\alpha}}$$

for all $x = (x', x''), y = (y', y'') \in \mathbb{R}^{N+1} \times \mathbb{R}$ with $|x''-y''| \geq \frac{1}{2}\max\{|x'|, |y'|\}$.

Proof. Let $\hat{x} = (w, x'')$ and $\hat{y} = (w, y'')$ where $w \in \mathbb{R}^{N+1}$ satisfies $|w| = 3|x-y|$. Note that $|x-y| \simeq |\hat{x}-x| \simeq |\hat{x}-\hat{y}|$. By (4.1) we have that $|F_\alpha(\hat{x}) - F_\alpha(\hat{y})| \simeq |w|^{-\frac{\alpha}{1+\alpha}}|x-y|$ and applying the fact that F_α is quasisymmetric twice (to the points $x, y, \hat{x}$ and to the points $\hat{x}, x, \hat{y}$), $|F_\alpha(x) - F_\alpha(y)| \simeq |F_\alpha(\hat{x}) - F_\alpha(\hat{y})| \simeq |x-y|^{\frac{1}{1+\alpha}}$. $\square$

The rest of this section is devoted to the proof of Proposition 4.1. As mentioned in Section 1.1, the construction in [22] can be used to deduce Proposition 4.1 for all rational $\alpha \in [N, N+1)$ but with no control on λ and η. For this reason, while in [22] the mapping F_α, for $\alpha = 1$, is obtained by iterating one family of bi-Lipschitz mappings $\Theta^{\mathcal{F}}$, here F_α is obtained by a periodic iteration of 2 families of bi-Lipschitz mappings $\Theta^{\mathcal{F}}_z$ (for the two values $z \in \{0, N + \frac{n-1}{n}\}$) in the alternating fashion of Section 4.1.

For the rest we fix integers $N \geq 0$, $n \geq 1$ and a rational number $\alpha \in [N, p_{N,n}]$ where, as before, $p_{N,n} = N + \frac{n-1}{n}$. In Section 4.2 we define the map F_α on the block Q and in Section 4.3 we extend it in all $\mathbb{R}^{N+2}$.

4.1. **A preliminary arrangement.** Suppose that $\frac{\alpha}{p_{N,n}} = \frac{a}{a+b}$ for some $a, b \in \mathbb{N}$. Using the next lemma we create a periodic sequence $(z_k)_{k\geq 1}$ that takes only the two values $0, p_{N,n}$ and $|z_1 + \cdots + z_k - k\alpha| \leq p_{N,n}$ for all $k \geq 1$.

Lemma 4.3. *Suppose that $y < z < x$ are such that $(a+b)z = ax + by$ for some nonnegative integers a, b. Then, there exists a finite sequence $(z_k)_1^{a+b}$ which has a terms x and b terms y such that, for all $k = 1, \ldots, a+b$,*

$$|z_1 + \cdots + z_k - kz| \leq x - y. \tag{4.2}$$

Proof. We may assume that $a, b \neq 0$; otherwise the claim is immediate.

Define z_k inductively as follows. Set $z_1 = x$. Suppose that the terms $z_1, \ldots, z_k$ have been defined; if $z_1 + \cdots + z_k \geq kz$, set $z_{k+1} = y$ and if $z_1 + \ldots z_k < kz$, set $z_{k+1} = x$.

Suppose that for some $k_0 < a+b$, the sequence $\{z_1, \ldots, z_{k_0}\}$ contains exactly b terms y. Then, $z_1 + \cdots + z_{k_0} - k_0 z = (a+b-k_0)(z-x) < 0$ and thus, $z_{k_0+1} = x$. Similarly, $z_k = x$ for all $k = k_0+1, \ldots, a+b$ and $(z_k)_1^{a+b}$ has exactly b terms y and a terms x. The same arguments apply if, for some $k_0 < a+b$, the sequence $\{z_1, \ldots, z_{k_0}\}$ contains exactly a terms x.

To show (4.2) we apply induction on k. If $k = 1$ then $|z_1 - z| = |x - z| < x - y$. Suppose that (4.2) is true for some $k < a+b$. Without loss of generality, assume that $z_1 + \cdots + z_k - kz \geq 0$. Then, $z_{k+1} = y$ and

$$y - x < y - z \leq z_1 + \cdots + z_{k+1} - (k+1)z \leq (x-y) + (y-z) \leq x - y. \qquad \square$$

By Lemma 4.3, there exists a sequence $(z_k)_1^{a+b}$ having a terms $p_{N,n}$ and b terms 0 such that, for each $k = 1, \ldots, a+b$,

$$|z_1 + \cdots + z_k - k\alpha| \leq p_{N,n}.$$

Extend z_k to all $k \in \mathbb{N}$ with $z_k = z_{k'}$ if $k \equiv k' \mod (a+b)$.

4.2. **A quasiconformal map on Q.** We define a quasiconformal mapping $f : (\mathsf{Q}, \mathsf{e}) \to (Q_I, e_{\mathcal{F}_0})$, iterating the mappings $\Theta^{\mathcal{F}}_z$ as in [22, Section 2.6].

Set $\mathsf{K}_0 = \mathsf{t}(\mathsf{Q})$ and for $k \geq 1$,

$$\mathsf{K}_{-k} = (M^{-k-z_1-\cdots-z_k}\mathfrak{C}^{N+1}) \times [0,1].$$

Moreover, define $\mathsf{T}_{-k} = \overline{\mathsf{K}_{-k} \setminus \mathsf{K}_{-k-1}}$ with $k \geq 0$. Then

$$\mathsf{Q} = (\{0\} \times [0,1]) \cup \bigcup_{k\geq 0} \mathsf{T}_{-k}.$$

Set also $K_0 = Q_I$, $K_{-1} = \kappa_{z_1}(Q_I)$, and $T_0 = \overline{K_0 \setminus K_{-1}} = \tau_{z_1}(Q)$. Let $f|\mathsf{T}_0 = \Theta_{z_1}^{\mathcal{F}_0} : \mathsf{T}_0 \to T_0$ noticing that $\mathsf{T}_0 = \mathsf{t}_{z_1}(\mathsf{Q})$.

For every $l \in [1, M^{1+z_1}]$, let ε_l be the marked flag-edge on Q_l derived from $(Q_I, e_{\mathcal{F}_0})$, and let $\sigma_{z_1}^l : (Q(l), e_{\mathcal{F}(l)}) \to (Q_l, \varepsilon_l)$ be the similarity mapping defined in Section 3.6 where $Q(l) \in \{Q_I, Q_L\}$ and $\mathcal{F}(l)$ is a $(N-1)$-flag of $\mathfrak{C}^{N+1}$. The similarity $\sigma_{z_1}^l$ induces naturally a core $\kappa_{z_2}(Q_l)$, consequently a tube $\tau_{z_2}(Q_l) = \overline{Q_l \setminus \kappa_{z_2}(Q_l)}$ to each block Q_l in K_{-1}.

Set $K_{-2} = \bigcup_l \kappa_{z_2}(Q_l)$ and $T_{-1} = \overline{K_{-1} \setminus K_{-2}} = \bigcup_l \tau_{z_2}(Q_l)$. Since $\mathsf{T}_{-1} = \bigcup_l \mathsf{t}_l$, the mapping $f|\mathsf{T}_{-1} : \mathsf{T}_{-1} \to T_{-1}$ is defined by gluing together homeomorphisms

$$f|\mathsf{t}_l = \sigma_{z_1}^l \circ \Theta_{z_2}^{\mathcal{F}(l)} \circ (\zeta_{z_1}^l)^{-1} : \mathsf{t}_l \to \tau_{z_2}(Q_l)$$

where $\zeta_{z_1}^l : (\mathsf{Q}, \mathsf{e}) \to (\mathsf{Q}_l, \mathsf{w} \cap \mathsf{Q}_l)$ is the similarity defined in Section 3.6.

The union W_{-1} of marked flag-edges ε_l is a flag-path along $\kappa_{z_2}(Q_I)$ going from $\{(\frac{M-2}{M^2}\mathcal{C}^j) \times \{0\} : \mathcal{C}^j \in \mathcal{F}_0\}$ to $\{(\frac{M-2}{M^2}\mathcal{C}^j) \times \{1\} : \mathcal{C}^j \in \mathcal{F}_0\}$, and the restrictions of $f|\mathsf{t}_l$ to the entrance and to the exit of t_l are identical modulo isometries for all l. Hence, we conclude that the gluing, therefore the homeomorphism $f|\mathsf{T}_{-1}$, is well-defined. We now have the extension $f : \mathsf{T}_0 \cup \mathsf{T}_{-1} \to T_0 \cup T_{-1}$.

For the next step, the index l in the previous step is replaced by l_1.

Fix $l_1 \in \{1, \dots, M^{1+z_1}\}$. Associated to each of the M^{1+z_2} blocks Q_{l_1,l_2} $(1 \le l_2 \le M^{1+z_2})$ in the core $\kappa_{l_1} = \kappa_{z_2}(Q_{l_1})$, the process of defining $f|\mathsf{t}_{l_1}$ has uniquely defined a core $\kappa_{l_1,l_2} = \kappa_{z_3}(Q_{l_1,l_2})$, a tube $\tau_{l_1,l_2} = \tau_{z_3}(Q_{l_1,l_2})$, a marked flag-edge ε_{l_1,l_2}, a block $Q(l_1,l_2) \in \{Q_I, Q_J\}$, an $(N-1)$-flag $\mathcal{F}(l_1,l_2)$ of $\mathfrak{C}^{N+1}$, and a similarity mapping

$$\sigma_{z_1,z_2}^{l_1,l_2} : (Q(l_1,l_2), e_{\mathcal{F}(l_1,l_2)}) \to (Q_{l_1,l_2}, \varepsilon_{l_1,l_2}).$$

Similarly, we define for each $l_2 = 1, \dots, M^{1+z_2}$ a similarity mapping

$$\zeta_{z_1,z_2}^{l_1,l_2} : (\mathsf{Q}, \mathsf{e}) \to (\mathsf{Q}_{l_1,l_2}, \mathsf{w} \cap \mathsf{Q}_{l_1,l_2}).$$

The union W_{-2} of these $M^{2+z_1+z_2}$ marked flag-edges is a flag-path along K_{-2} from $\{(\frac{M-2}{M^3}\mathcal{C}^j) \times \{0\} : \mathcal{C}^j \in \mathcal{F}_0\}$ to $\{(\frac{M-2}{M^3}\mathcal{C}^j) \times \{1\} : \mathcal{C}^j \in \mathcal{F}_0\}$, and the union K_{-3} of the cores of these $M^{2+z_1+z_2}$ new blocks is a topological $(N+2)$-cube. Set $T_{-2} = \overline{K_{-2} \setminus K_{-3}}$. We now extend $f : \mathsf{T}_0 \cup \mathsf{T}_{-1} \cup \mathsf{T}_{-2} \to T_0 \cup T_{-1} \cup T_{-2}$ by gluing together the homeomorphisms

$$f|\mathsf{t}_{l_1,l_2} = \sigma_{z_1,z_2}^{l_1,l_2} \circ \Theta_{z_3}^{\mathcal{F}(l_1,l_2)} \circ (\zeta_{z_1,z_2}^{l_1,l_2})^{-1}|\mathsf{t}_{l_1,l_2} \to \tau_{l_1,l_2}.$$

Continuing this process inductively in a self-similar manner, we obtain a homeomorphism $f : \mathsf{Q} \setminus (\{0\} \times [0,1]) \to Q_I \setminus \gamma$, where γ is the snowflake open curve $\gamma = \bigcap_{k=1}^{\infty} K_{-k}$.

Lemma 4.4. *There exists $C > 1$ depending only on N, n such that $M^{-\alpha k} f$ is C-bi-Lipschitz on each of the $M^{k+z_1+\cdots+z_k}$ tubes in T_{-k}.*

Proof. The scaling factor of each $\zeta_{z_1,\dots,z_k}^{l_1,\dots,l_k}$ is $M^{-k-z_1-\cdots-z_k}$ and the scaling factor of each $\sigma_{z_1,\dots,z_k}^{l_1,\dots,l_k}$ is $\frac{M-2}{M}M^{-k}$. Moreover, only a finite number of different bi-Lipschitz mappings $\Theta_z^{\mathcal{F}}$ have been used in the definition of f. Therefore, by Lemma 4.3, $M^{-\alpha k} f$ is C-bi-Lipschitz on each of the $M^{k+z_1+\cdots+z_k}$ tubes in T_{-k}, for some constant $C > 1$ depending on M, $p_{N,n}$, and the bi-Lipschitz constants of the maps $\Theta_0^{\mathcal{F}}, \Theta_{p_{N,n}}^{\mathcal{F}}$; thus C depends only on N, n. $\square$

Hence, the mapping $f : \mathsf{Q} \setminus (\{0\} \times [0,1]) \to Q_I \setminus \gamma$ is K-quasiconformal for some K depending only on N, n. By a theorem of Väisälä for removable singularities [19, Theorem 35.1], f can be extended to a K-quasiconformal mapping from Q onto Q_I.

Remark 4.5. *Note the following self-similar property on I-blocks: whenever $Q_{l_1,\dots,l_{a+b}}$ is an I-block of K_{-a-b} then*

$$f|\mathsf{t}_{l_1,\dots,l_{a+b}} = \sigma^{l_1,\dots,l_{a+b}}_{z_1,\dots,z_{a+b}} \circ f|\mathsf{t} \circ (\zeta^{l_1,\dots,l_{a+b}}_{z_1,\dots z_{a+b}})^{-1}|\mathsf{t}_{l_1,\dots,l_{a+b}}. \tag{4.3}$$

In particular, the periodicity of $\{z_k\}$ with period $a+b$ implies the periodicity of f (up to similarities) to tubes $\mathsf{t}_{l_1,\dots,l_k}$ and $\mathsf{t}_{l_1,\dots,l_{k+a+b}}$ when $Q_{l_1,\dots,l_k}$, $Q_{l_1,\dots,l_{k+a+b}}$ are I-blocks.

Finally, note that the snowflake curve $\gamma = \bigcup_{k=1}^{\infty} K_{-k}$ is the image of the line segment $\{0\} \times [0,1]$ under f.

4.3. **Quasiconformal extension to $\mathbb{R}^{N+2}$.** We now extend the mapping $f : \mathsf{Q} \to Q_I$ to a quasiconformal homeomorphism of $\mathbb{R}^{N+2}$ by backward iteration.

Fix an I-block $Q_{l_1,\dots,l_{a+b}}$ in some core $\kappa_{l_1,\dots,l_{a+b}}$ of Q_I with $l_i \neq 1, M^{1+z_i}$. Such a block exists by the first property of the path J_I in Section 3.3.

Let $\zeta = \zeta^{l_1,\dots,l_{a+b}}_{z_1,\dots,z_{a+b}}$ be the similarity in $\mathbb{R}^{N+2}$ that maps (Q, e) to $(\mathsf{Q}_{l_1,\dots,l_{a+b}}, \mathsf{w} \cap \mathsf{Q}_{l_1,\dots,l_{a+b}})$, and $\sigma = \sigma^{l_1,\dots,l_{a+b}}_{z_1,\dots,z_{a+b}}$ be the similarity in $\mathbb{R}^{N+2}$ that maps $(Q_I, e_{\mathcal{F}_0})$ to $(Q_{l_1,\dots,l_{a+b}}, w_{\mathcal{F}_0} \cap Q_{l_1,\dots,l_{a+b}})$ as in Section 4.2. Note that ζ has a scaling factor $M^{-(a+b)(1+\alpha)}$ and σ has a scaling factor $\frac{M-2}{M} M^{-(a+b)}$.

Because $l_i \neq 1, M^{1+z_i}$, the space $\mathbb{R}^{N+2}$ is the union of an increasing sequence of I-blocks and regular blocks

$$\mathbb{R}^{N+2} = \bigcup_{k\geq 0} \sigma^{-k} Q_I = \bigcup_{k \geq 0} \zeta^{-k}\mathsf{Q}.$$

If $l_i = 1$ for all $i = 1, \dots, a+b$ or $l_i = M^{1+z_i}$ for all $i = 1, \dots, a+b$ then these unions would be proper subsets of $\mathbb{R}^{N+2}$.

Define homeomorphisms $F^{(k)} : \zeta^{-k}\mathsf{Q} \to \sigma^{-k} Q_I$, $k \geq 0$, by

$$F^{(k)} = \sigma^{-k} \circ f \circ \zeta^k. \tag{4.4}$$

The self similar property (4.3) implies that $f \circ \zeta|\mathsf{Q} = \sigma \circ f$. Therefore, $F^{(k)}|\mathsf{Q} = \sigma^{-k} \circ f \circ \zeta^k|\mathsf{Q} = f$ for all $k \geq 0$, and $F^{(k')}|\zeta^{-k}\mathsf{Q} = F^{(k)}$ for all $k' \geq k \geq 0$. Thus, the mapping $F_\alpha = \lim_{k\to\infty} F^{(k)} : \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$ is well-defined. Moreover, since all mappings $F^{(k)}$ are K-quasiconformal, F_α is K-quasiconformal and therefore, F_α is η-quasisymmetric for some η depending only on N, n.

Remark 4.6. *The backward iteration depends on the fact that α is rational. In fact, for any real number $\alpha \in [N, p_{N,n}]$, the arguments of Lemma 4.3 can be used to find a sequence $(z_k)_{k\geq 0}$ having terms in $\{0, p_{N,n}\}$ such that $|z_1 + \dots + z_k - k\alpha| \leq p_{N,n}$ for all $k \geq 0$. Therefore, a quasiconformal map $f : \mathsf{Q} \to Q_I$ can be constructed as in Section 4.2. However, if α is irrational the sequence (z_k) is not periodic and the backward iteration cannot be used to extend this map in all $\mathbb{R}^{N+2}$.*

We show now that the quasisymmetric mapping F_α satisfies (4.1).

Proof of Proposition 4.1. By the self similar property (4.4) and the scaling factors of ζ, σ, it is enough to show (4.1) only for $x = (x', x'') \in \mathsf{Q}$ with $|x'| \neq 0$. Suppose that $x \in t_{l_1 \cdots l_k}$. Then, $B(x, \frac{1}{2}|x'|)$ intersects at most m annulus tubes $t_{l_1 \cdots l_{k'}}$ for some m depending only on M, thus on N, n. Since $|x'| \simeq M^{-k-z_1-\cdots-z_k} \simeq M^{-k(\alpha+1)}$, we deduce (4.1) by Lemma 4.4. □

5. Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. Using Proposition 4.1, we first show in Section 5.1 the theorem when $\alpha \geq 0$ is a rational number and then, in Section 5.2, we prove the theorem for all real numbers $\alpha \geq 0$ applying an Arzelà-Ascoli limiting argument.

Assuming Theorem 1.1, the proof of Corollary 1.3 is as follows.

Proof of Corollary 1.3. Suppose that $\alpha \in [0, 1)$ and g is a bi-Lipschitz embedding of the singular line $\Gamma = \{x_1 = 0\} \subset \mathbb{G}_\alpha$ into $\mathbb{R}^2$. We show that g extends to a bi-Lipschitz embedding of $\mathbb{G}_\alpha$ onto $\mathbb{R}^2$.

Let $f \colon \mathbb{G}_\alpha \to \mathbb{R}^2$ be the bi-Lipschitz mapping of Theorem 1.1. Then, $g(\Gamma)$ and $f(\Gamma)$ are quasilines in $\mathbb{R}^2$ and $g \circ f^{-1}$ is a bi-Lipschitz homeomorphism between these quasilines. Consider an η-quasisymmetric mapping $h : \mathbb{R} \to f(\Gamma)$. By the Beurling-Ahlfors quasiconformal extension [7], there exists a K-quasiconformal extension $F : \mathbb{R}^2 \to \mathbb{R}^2$ of h, with K depending only on η that satisfies

$$\operatorname{diam} F(I) \simeq |DF(x)| \operatorname{diam} I$$

for every arc $I \subset \mathbb{R} \times \{0\}$ and every point $x \in \mathbb{R}^2$ for which $\operatorname{dist}(x, I) \simeq |I|$. Here the ratio constants depend only on η. Similarly, there exists a quasiconformal mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ extending $g \circ f^{-1} \circ h$ satisfying the properties of F.

We claim that $F = G \circ F^{-1} \circ f$ is bi-Lipschitz extension of g. Indeed, for any point $x \in \mathbb{R}^2$ we have $|DF(x)|/|DG(x)| \simeq \operatorname{diam} F(I)/\operatorname{diam} G(I)$ for some suitable $I \subset \mathbb{R} \times \{0\}$. Since $g \circ f^{-1}$ is bi-Lipschitz, the last ratio is comparable to 1. Therefore, $|DF(x)| \simeq |DG(x)|$ and $G \circ F^{-1}$ is bi-Lipschitz. □

5.1. **Proof of Theorem 1.1 when α is rational.**

We first recall two basic properties of the generalized Grushin metric.

The dilation property states that for any $\alpha \geq 0$ and any $\delta > 0$,

$$d_{\mathbb{G}_\alpha}((\delta x_1, \delta^{1+\alpha} x_2), (\delta y_1, \delta^{1+\alpha} y_2)) = \delta d_{\mathbb{G}_\alpha}((x_1, x_2), (y_1, y_2)).$$

This can be found in [6] for the case $\alpha = 1$, but it applies equally to the case of arbitrary $\alpha \geq 0$.

Given $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{G}_\alpha$ we define the *Grushin quasidistance*

$$d_\alpha(x, y) = |x_1 - y_1| + \min\left\{ |x_2 - y_2|^{\frac{1}{1+\alpha}}, \frac{|x_2 - y_2|}{|x_1|^\alpha} \right\}.$$

It is well-known that the quasidistance $d_\alpha(x, y)$ is comparable to $d_{\mathbb{G}_\alpha}(x, y)$; see e.g. [11, Theorem 2.6]. In fact, the following result holds true.

Lemma 5.1. *For each $m \geq 0$ there exists $C(m) > 1$ such that for all $\alpha \in [0, m]$ and all $x, y \in \mathbb{G}_\alpha$*

$$C(m)^{-1} d_\alpha(x, y) \leq d_{\mathbb{G}_\alpha}(x, y) \leq C(m) d_\alpha(x, y)$$

The proof of Lemma 5.1 is identical to that of Lemma 2 in [1]. The next lemma is a simple application of the Mean Value Theorem and its proof is left to the reader.

Lemma 5.2. *For all $m \geq 0$ there exists $c(m) > 1$ such that for all $\alpha \in [0, m]$ and $x, y \in \mathbb{R}$ with $|x| \geq |y|$,*

$$c(m)^{-1}|x|^{\alpha}|x - y| \leq |x|x|^{\alpha} - y|y|^{\alpha}| \leq c(m)|x|^{\alpha}|x - y|.$$

For each number $\alpha \in [N, N + \frac{n-1}{n}]$ define $H_\alpha : \mathbb{G}_\alpha \to \mathbb{R} \times \{0\} \times \mathbb{R} \subset \mathbb{R}^{N+2}$ to be the mapping

$$H_\alpha(x_1, x_2) = (x_1|x_1|^{\alpha}, 0, \dots, 0, x_2).$$

It is known that H_α is an η'-quasisymmetric mapping with η' depending only on N, n; see e.g. [1, Theorem 2].

We are now ready to prove Theorem 1.1 for rational $\alpha \geq 0$. The argument in this case is analogous to those of [22, Theorem 1.1] and [23, Theorem 5.1].

Proposition 5.3. *For all integers $N \geq 0$ and $n \geq 1$, there exists $L > 1$ depending only on N, n such that, for each rational $\alpha \in [N, N + \frac{n-1}{n}]$, there exists an L-bi-Lipschitz homeomorphism of $\mathbb{G}_\alpha$ onto a 2-dimensional quasiplane $\mathcal{P}_\alpha \subset \mathbb{R}^{N+2}$.*

Proof. Fix a rational $\alpha \in [N, N + \frac{n-1}{n}]$ and let λ and $F_\alpha : \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$ be the constant and η-quasisymmetric map, respectively, of Proposition 4.1 with λ and η depending only on N, n. The composition $F_\alpha \circ H_\alpha$ is a homeomorphism from $\mathbb{G}_\alpha$ onto the quasiplane $\mathcal{P}_\alpha = F_\alpha(\mathbb{R} \times \{0\} \times \mathbb{R})$. We show that $F_\alpha \circ H_\alpha$ is L-bi-Lipschitz with L depending only on λ, the quasisymmetric data η, η' of F_α, H_α, respectively, the constant $C(N + \frac{n-1}{n})$ of Lemma 5.1 and the constant $c(N + \frac{n-1}{n})$ of Lemma 5.2; thus L depends only on N, n. The comparison constants below depend at most on N, n.

Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be points in $\mathbb{G}_\alpha$ and assume that $|x_1| \geq |y_1|$. The proof splits into four cases.

Case I. $|x_1| > 0$, $|x_1 - y_1| \leq |x_1|/4$, and $|x_2 - y_2| \leq |x_1|^{1+\alpha}/2$. Then, $|x_1| \simeq |y_1|$ and the Grushin distance satisfies $d_{\mathbb{G}_\alpha}(x, y) \simeq |x_1 - y_1| + |x_1|^{-\alpha}|x_2 - y_2|$. Moreover, by Lemma 5.2, $|H_\alpha(x) - H_\alpha(y)| \simeq |x_1|^{\alpha}|x_1 - y_1| + |x_2 - y_2|$.

If $H_\alpha(y) \in B(H_\alpha(x), \frac{1}{2}|x_1|^{1+\alpha})$ then Proposition 4.1 yields $|F_\alpha \circ H_\alpha(x) - F_\alpha \circ H_\alpha(y)| \simeq |x_1 - y_1| + |x_1|^{-\alpha}|x_2 - y_2| \simeq d_{\mathbb{G}_\alpha}(x, y)$.

Otherwise, $|H_\alpha(x) - H_\alpha(y)| \simeq |x_1|^{1+\alpha}$. Let $z \in \mathbb{G}_\alpha$ such that $|H_\alpha(x) - H_\alpha(z)| = |x_1|^{1+\alpha}/2$. Then the quasisymmetry of F_α and H_α implies $|F_\alpha \circ H_\alpha(x) - F_\alpha \circ H_\alpha(y)| \simeq |F_\alpha \circ H_\alpha(x) - F_\alpha \circ H_\alpha(z)| \simeq d_{\mathbb{G}_\alpha}(x, z) \simeq d_{\mathbb{G}_\alpha}(x, y)$.

Case II. $|x_1| > 0$, $|x_1 - y_1| \geq |x_1|/4$, and $|x_2 - y_2| \leq |x_1|^{1+\alpha}/2$. Then, $d_{\mathbb{G}_\alpha}(x, y) \simeq |x_1 - y_1| \simeq |x_1|$ and, by Lemma 5.2, $|H_\alpha(x) - H_\alpha(y)| \simeq |x_1|^{1+\alpha}$. Similar to the second part of Case I, $|F_\alpha \circ H_\alpha(x) - F_\alpha \circ H_\alpha(y)| \simeq d_{\mathbb{G}_\alpha}(x, y)$.

Case III. $|x_1| > 0$ and $|x_2 - y_2| \geq |x_1|^{1+\alpha}/2$. Then, $d_{\mathbb{G}_\alpha}(x, y) \simeq |x_1 - y_1| + |x_2 - y_2|^{1/(1+\alpha)} \simeq |x_2 - y_2|^{1/(1+\alpha)}$. By Corollary 4.2, $|F_\alpha \circ H_\alpha(x) - F_\alpha \circ H_\alpha(y)| \simeq |H_\alpha(x) - H_\alpha(y)|^{\frac{1}{1+\alpha}} \simeq |x_2 - y_2|^{\frac{1}{1+\alpha}} \simeq d_{\mathbb{G}_\alpha}(x, y)$.

Case IV. $x_1 = 0$. Then, $|F_\alpha \circ H_\alpha(x) - F_\alpha \circ H_\alpha(y)| \simeq d_{\mathbb{G}_\alpha}(x, y)$ by taking limits in Case III. $\square$

5.2. **Proof of Theorem 1.1 when α is irrational.** The following lemma deals with the bi-Lipschitz embeddability of $\mathbb{G}_\alpha$ into $\mathbb{R}^{[\alpha]+2}$ for all real $\alpha \geq 0$.

Lemma 5.4. *For all integers $N \geq 0$ and $n \geq 1$, there exists $L > 1$ depending only on N, n such that for all $\alpha \in [N, N + \frac{n-1}{n}]$ there exists an L-bi-Lipschitz embedding $f_\alpha : \mathbb{G}_\alpha \to \mathbb{R}^{N+2}$.*

Proof. Fix a number $\alpha \in [N, N + \frac{n-1}{n}]$ and let $(q_k)_{k\in\mathbb{N}}$ be a sequence of rational numbers in $[N, N + \frac{n-1}{n}]$ converging to α. Note that $\lim_{k\to\infty} d_{\mathbb{G}_{q_k}}(x, y) = d_{\mathbb{G}_\alpha}(x, y)$ for each $x, y \in \mathbb{R}^2$. By Proposition 5.3, there exists $L > 1$ depending only on N, n such that, for each q_k, there is an L-bi-Lipschitz map $f_{q_k} : \mathbb{G}_{q_k} \to \mathbb{R}^{N+2}$. It is clear by their construction that each f_{q_k} maps $(0, 0)$ to $(0, \dots, 0) \in \mathbb{R}^{N+2}$.

Let $\mathscr{A} = \{a_1, a_2, \dots\}$ be a countable dense set in $(\mathbb{G}_\alpha, d_{\mathbb{G}_\alpha})$. Note that, for each $i \in \mathbb{N}$, $|f_{q_k}(a_i)| \leq L d_{\mathbb{G}_{q_k}}(a_i, (0, 0))$. Hence, for each $i \in \mathbb{N}$, the sequence $(f_{q_k}(a_i))_{k\in\mathbb{N}}$ is bounded. Define, for each $i \in \mathbb{N}$, a subsequence of $(f_{q_k})_{k\in\mathbb{N}}$ as follows. Set $(f_k^0)_{k\in\mathbb{N}} = (f_{q_k})_{k\in\mathbb{N}}$ and for each $i \in \mathbb{N}$ let $(f_k^i)_{k\in\mathbb{N}}$ be a subsequence of $(f_k^{i-1})_{k\in\mathbb{N}}$ so that $(f_k^i(a_i))_{k\in\mathbb{N}}$ converges. Then, for each $a_i \in \mathscr{A}$, the sequence $(f_k^k(a_i))_{k\in\mathbb{N}}$ converges. Set $f(a_i) = \lim_{k\to\infty} f_k^k(a_i)$.

We claim that $f : (\mathscr{A}, d_{\mathbb{G}_\alpha}) \to \mathbb{R}^{N+2}$ is L-bi-Lipschitz. Let $z_1, z_2 \in \mathscr{A}$ and $\epsilon > 0$. Choose $k \in \mathbb{N}$ big enough so that

$$|f_k^k(z_i) - f(z_i)| \leq \frac{\epsilon}{3} \quad \text{for each } i = 1, 2 \tag{5.1}$$

and if $f_k^k = f_{q(k)}$ for some $q(k) \in \{q_1, q_2, \dots\}$ then

$$|d_{\mathbb{G}_{q(k)}}(z_1, z_2) - d_{\mathbb{G}_\alpha}(z_1, z_2)| \leq \frac{\epsilon}{3L}. \tag{5.2}$$

Combining (5.1) and (5.2) we have that

$$|f(z_1) - f(z_2)| \leq L d_{\mathbb{G}_\alpha}(z_1, z_2) + \epsilon.$$

Similarly, $|f(z_1) - f(z_2)| \geq \frac{1}{L} d_{\mathbb{G}_\alpha}(z_1, z_2) - \epsilon$. Since ϵ is arbitrary, the claim follows.

Using the density of $\mathscr{A}$ in $\mathbb{G}_\alpha$, the mapping f can be extended to all $\mathbb{G}_\alpha$ uniquely. It remains to show that $f : \mathbb{G}_\alpha \to \mathbb{R}^{N+2}$ is L-bi-Lipschitz. Let $x_1, x_2 \in \mathbb{G}_\alpha$ and $\epsilon > 0$. Find $z_1, z_2 \in \mathscr{A}$ such that for each $i = 1, 2$, $d_{\mathbb{G}_\alpha}(x_i, z_i) < \frac{\epsilon}{4L}$ and $|f(x_i) - f(z_i)| < \frac{\epsilon}{4}$. Then,

$$|f(x_1) - f(x_2)| \leq L d_{\mathbb{G}_\alpha}(z_1, z_2) + \frac{\epsilon}{2} \leq L d_{\mathbb{G}_\alpha}(x_1, x_2) + \epsilon.$$

Similarly, $|f(x_1) - f(x_2)| \geq \frac{1}{L} d_{\mathbb{G}_\alpha}(x_1, x_2) - \epsilon$. Since ϵ is arbitrary, f is L-bi-Lipschitz. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. Let $\alpha \in [N, N + \frac{n-1}{n}]$ and (q_k) be a sequence of rationals in $[N, N + \frac{n-1}{n}]$ converging to α such that the L-bi-Lipschitz maps $f_{q_k} = F_{q_k} \circ H_{q_k}$ converge to an L-bi-Lipschitz map $f_\alpha : \mathbb{G}_\alpha \to \mathbb{R}^{N+2}$ as in the proof of Lemma 5.4. Here $F_{q_k} : \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$ is the quasisymmetric mapping of Proposition 4.1, $H_{q_k}(x, y) = (x|x|^{q_k}, 0, \dots, 0, y)$ is the quasisymmetric mapping of $\mathbb{G}_{q_k}$ onto $\mathbb{R}^2$ and L depends only on N, n. Note that the mappings H_{q_k} converge pointwise to the mapping $H_\alpha = (x|x|^\alpha, 0, \dots, 0, y)$

and that the mappings F_{q_k} fix the origin of $\mathbb{R}^{N+2}$ and the vector $(0,\dots,0,1)$. By [12, Corollary 10.30], passing to a subsequence, we may assume that F_{q_k} converges to a quasisymmetric mapping F_α. Then, $f_\alpha = H_\alpha \circ F_\alpha$, and the image of f_α is $F_\alpha(\mathbb{R}\times\{0\}\times\mathbb{R})$ which is a 2-dimensional quasiplane in $\mathbb{R}^{N+2}$. □

6. Appendix

This section gives the construction of the paths $J_I(N,n), J_L(N,n)$ used in Section 3.3 and the proof of Lemma 3.1. In Section 6.1, we construct for each integer $N \geq 0$ and each integer $M = 4k+5 \geq 9$ paths $\mathcal{J}_I^N(M)$, $\mathcal{J}_I^N(M)$ which serve as a base for the construction of paths $J_I(N,n), J_L(N,n)$ in Section 6.2. Then, in Section 6.3 we show Lemma 3.1.

6.1. **Auxiliary paths.** Let $N \geq 0$ and $M = 4k+5 \geq 9$ be integers. The paths $\mathcal{J}_I^N(M), \mathcal{J}_I^N(M)$ are defined by induction on N.

For an integer $M = 4k+5 \geq 9$ let $\mathcal{J}_I^0(M)$ be the segment $I \subset \mathbb{R}^2$ which we divide into M disjoint I-segments ℓ_m of length $1/M$. Similarly, let $\mathcal{J}_L^0(M)$ be the segment $L \subset \mathbb{R}^2$ which we divide into M disjoint I- and L-segments ℓ_m of length $1/M$ where $\ell_{\frac{M-1}{2}}$ is an L-segment and the rest are I-segments.

To obtain $\mathcal{J}_I^1(M)$, replace each pair of I-segments $\ell_m \cup \ell_{m+1}$, where $m \in \{2,4,\dots,\frac{M-5}{2},\frac{M+5}{2},\dots,M-4,M-2\}$, by a swath containing $\frac{M-1}{2}$ I- and L-segments of length $1/M$ running in the negative x_1-direction; see Figure 1 for a schematic representation. Precisely, $\mathcal{J}_I^1(M)$ contains $\frac{M-5}{2}$ swaths and each swath contains 4 L-segments and $\frac{M-5}{2}$ pairs of consecutive I-segments. Here we make use of the fact that $M = 4k+5$.

To obtain $\mathcal{J}_I^2(M)$ replace each of the $(M-5)^2/4$ pairs of consecutive I-segments in $\mathcal{J}_I^2(M)$ by a swath containing $\frac{M-1}{2}$ many I- and L-segments of length $1/M$ running in the positive x_3-direction; see Figure 1. Note that $\mathcal{J}_I^2(M)$ contains $(M-5)^2/4$ new swaths, each containing $\frac{M-5}{2}$ pairs of consecutive I-segments.

Proceeding inductively, we obtain for each integer $N \geq 0$ and each integer $M = 4k+5 \geq 9$ a path $\mathcal{J}_I^N(M)$. Denote by $(\#\mathcal{J}_I^N(M))$ the total number of I- and L-segments in $\mathcal{J}_I^N(M)$, and by $(\#\mathcal{J}_I^N(M))^*$ the total number of pairs of consecutive I-segments. Then, $(\#\mathcal{J}_I^0(M)) = M$, $(\#\mathcal{J}_I^0(M))^* = \frac{M-5}{2}$ and for $N \geq 1$

$$\begin{aligned}(\#\mathcal{J}_I^N(M)) &= (\#\mathcal{J}_I^{N-1}(M)) + (\#\mathcal{J}_I^{N-1}(M))^*(M-3)\\ (\#\mathcal{J}_I^N(M))^* &= (\#\mathcal{J}_I^{N-1}(M))^*\frac{M-5}{2}.\end{aligned}$$

Therefore,

$$(\#\mathcal{J}_I^N(M)) = M + (M-3)(M-5)\frac{(M-5)^N - 2^N}{2^{N+1}(M-7)}$$

and

$$(\#\mathcal{J}_I^N(M))^* = \frac{(M-5)^{N+1}}{2^{N+1}}.$$

The paths $\mathcal{J}_L^N(M)$ are constructed similarly; see Figure 1.

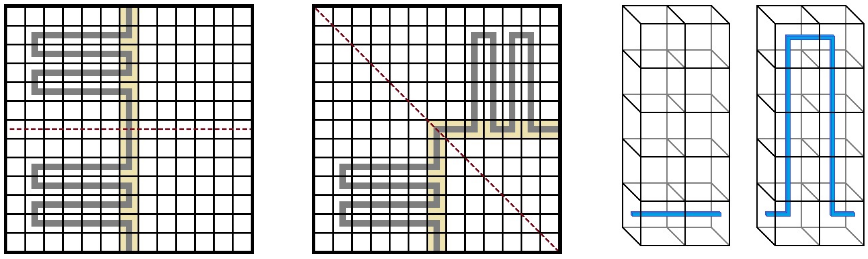

FIGURE 1. The paths $\mathcal{J}_I^0(M)$, $\mathcal{J}_L^0(M)$ and a swath in the extra dimension.

6.2. **Construction of the paths J_I, J_L.** Fix integers $N \geq 0$ and $n \geq 1$ and set $M = M_{N,n} = 9^{n(N+2)}$. We first construct paths $\tilde{J}_I(N,n)$ and $\tilde{J}_L(N,n)$ as an extension of $\mathcal{J}_I^N$ and $\mathcal{J}_L^N$, respectively, in an extra dimension. The required paths $J_I(N,n)$ and $J_L(N,n)$ are obtained after applying a suitable rotation to $\tilde{J}_I(N,n)$ and $\tilde{J}_L(N,n)$ respectively.

We work first for $\tilde{J}_I(N,n)$. To construct $\tilde{J}_I(N,n)$ we use the path $\mathcal{J}_I^N(M)$ which contains $M' = (\#\mathcal{J}_I^N(M))^* = 2^{-N-1}(M-5)^{N+1}$ pairs of consecutive I-segments. Replace each pair of I-segments $\ell_m \cup \ell_{m+1}$ in $\mathcal{J}_I^N(M)$, $m = 1,\dots,M'$, by a swath consisting of $2k_m+2$ many I- and L-segments, running in the positive x_{N+2}-direction. Here, $0 \leq k_m \leq \frac{M-3}{2}$ (if $k_m = 0$ then the swath contains only ℓ_m, ℓ_{m+1} and if $k_m \geq 1$ then it contains 4 L-segments and $2(k_m-1)$ I-segments). The resulting path is denoted by $\tilde{J}_I(N,n)$. Moreover we require that the swaths are chosen in such a way that $\tilde{J}_I(N,n)$ is symmetric with respect to the plane $x_{N+1} = 1/2$. Hence, for each $m \in \{1,\cdots,M'\}$ there is $m' \in \{1,\dots,M'\}$, $m \neq m'$ such that $k_m = k_{m'}$.

The path $\tilde{J}_I(N,n)$ must consist of $M^{1+p_{N,n}} = M^{N+2-1/n}$ many I- and L-segments of length $1/M$. Thus, we require that

$$2(k_1+k_2+\cdots+k_{M'}) + 2M' + ((\#\mathcal{J}_I^N(M)) - 2M') = M^{N+2-1/n}$$

or equivalently

$$k_1+k_2+\cdots+k_{M'} = \frac{M^{N+2-1/n} - (\#\mathcal{J}_I^N(M))}{2}. \tag{6.1}$$

The symmetry of $\tilde{J}_I(N,n)$ implies that the left hand side of (6.1) is even. Moreover, since M is a multiple of 9, the right hand side of (6.1) is also even. Since each k_m can take any integer value in $[0, M-3]$, the left hand side of (6.1) can take any even integer value in $[0, 2(M-3)M']$ and it is enough to show that

$$2(M-3)M' \geq \frac{M^{N+2-1/n} - (\#\mathcal{J}_I^N(M))}{2}.$$

Indeed, since $M = 9^{n(N+2)}$,

$$2(M-3)M' = \frac{(M-3)(M-5)^{N+1}}{2^N} \geq \left(\frac{M-5}{2}\right)^{N+2} \geq M^{N+2-\frac{1}{n}}.$$

Properties (1)–(4) of Section 3.3 are immediate. The proof of property (5) is almost identical to the proof of Lemma 3.1 in the following section.

The path $\tilde{J}_L(N,n)$ is obtained similarly. In this case we require symmetry with respect to the plane $x_1 + x_{N+1} = \frac{1}{2}$.

6.3. **Proof of Lemma 3.1.** We show Lemma 3.1 for $z = p$ and $Q = Q_I$. Similar arguments apply when $Q = Q_L$. For the rest, $\mathcal{F} = \mathcal{F}_0$.

By Section 6.2, each $J_I(N,n)$ is constructed as a sequence of paths $I = J_1, J_2, \dots, J_{N+2} = J_I(N,n)$ where each J_k lies in a k-dimensional subspace of $\mathbb{R}^{N+2}$ and J_{k+1} is constructed by replacing pairs of I-segments $\ell_m \cup \ell_{m+1}$ of J_k by swaths $\mathscr{S} = I_m \cup \mathscr{S}_m \cup I'_{m+1}$. Here, $I_m \subset \ell_m$, $I'_{m+1} \subset \ell_{m+1}$ are line segments and $\mathscr{S}_m$ is a polygonal arc perpendicular to the k-plane containing J_k. Associated to each J_k we consider a core $\kappa_k = \mathcal{T}^{N+2}(J_k, \frac{M-2}{M^2})$.

Each core κ_k consists of M_k many I- and L-blocks $Q_{k,m}$, $m = 1, \dots, M_k$. Here $M_k = (\#\mathcal{J}_I^k(M))$, if $k = 0, \dots, N+1$, and $M_{N+2} = M^{1+p_{N,n}}$ with $M = 9^{n(N+2)}$. Similar to the path J_k, each core κ_k is constructed by removing certain pairs of I-blocks from κ_{k-1} and replacing these pairs by solid swaths $\mathcal{S} = \mathcal{T}^{N+2}(\mathscr{S}, \frac{M-2}{M^2})$. Note that $\kappa_1 = \kappa_0(Q)$.

For each k, m the side $\mathrm{s}(Q_{k,m})$ has a fibration into I-segments (if $Q_{k,m}$ is an I-block) or L-segments (if $Q_{k,m}$ is an L-block) similar to the fibrations $\{I_x\}, \{L_x\}$ of Section 3.6. The fibrations of the sides of $Q_{k,m}$ induce a fibration of the side $\mathrm{s}(\kappa_k) = \bigcup_u \Gamma_{k,u}$ where $\Gamma_{k,u}$ is a polygonal arc, $u \in \partial\mathfrak{C}^{N+1}$ and $\Gamma_{k,u} \cap \mathrm{s}(Q_{k,m})$ is a fiber of $\mathrm{s}(Q_{k,m})$. As with the paths J_k, each $\Gamma_{k+1,u}$ is constructed by replacing certain line segments of $\Gamma_{k,u}$ by fibers which lie on the new solid swaths of κ_{k+1}. Note that if u is a vertex of $\mathfrak{C}^{N+1}$ then $\Gamma_{k,u}$ is an edge of κ_k and $\Gamma_{N+2,u}$ is an element of the flag-path $w_{\mathcal{F}_0}$ of $\kappa_p(Q_I)$.

For the construction of $\Theta_p^{\mathcal{F}_0}$ we first map $\tau_p(Q)$ onto $\tau_0(Q)$ and then we compose with $\Theta_0^{\mathcal{F}_0}$.

Step 1: We map (Q, κ_{N+2}) onto (Q, κ_1). We construct a bi-Lipschitz map in Q which fixes ∂Q and maps κ_{N+2} onto κ_{N+1} by compressing each solid swath onto the two I-blocks of κ_{N+1} which it replaced. The map is defined in a neighbourhood of each solid swath.

For each solid swath $\mathcal{S} \subset \kappa_{N+2}$, consider a $(N+2)$-box $\widetilde{Q}(\mathcal{S}) \subset Q$ which contains $\mathcal{S}$ and satisfies the following properties,

(1) each face of $\widetilde{Q}(\mathcal{S})$ is parallel to a coordinate $(N+1)$-hyperplane;
(2) $\widetilde{Q}(\mathcal{S}) \cap \kappa_{N+2} = \mathcal{S}$;
(3) $\widetilde{Q}(\mathcal{S})$ and $\widetilde{Q}(\mathcal{S}')$ have disjoint interiors if $\mathcal{S} \neq \mathcal{S}'$.

For each solid swath $\mathcal{S} \subset \kappa_{N+2}$ we construct a bi-Lipschitz isotopy $\Phi_{\mathcal{S}} : \widetilde{Q}(\mathcal{S}) \times I \to \widetilde{Q}(\mathcal{S})$ such that $\Phi_{\mathcal{S}}(\cdot, t)|\partial\widetilde{Q}(\mathcal{S}) = \mathrm{id}$ for all $t \in [0,1]$, $\Phi_{\mathcal{S}}(\cdot, 0) = \mathrm{id}$, and $\Phi_{\mathcal{S}}(\cdot, 1)|\mathcal{S}$ is a PL bi-Lipschitz map of $\mathcal{S}$ onto the two I-blocks of κ_{N+1} that $\mathcal{S}$ replaced. By PL bi-Lipschitz isotopy, we mean that the induced mapping $g_{t_1 t_2} = \Phi_{\mathcal{S}}(\Phi_{\mathcal{S}}^{-1}(\cdot, t_1), t_2)$ is piecewise linear and $(1 + C|t_2 - t_1|)$-bi-Lipschitz for some constant $C > 0$ and all $t_1, t_2 \in [0,1]$. Note that $g_{t_1 t_2}^{-1} = g_{t_2 t_1}$.

Fix a solid swath $\mathcal{S} \subset \kappa_{N+2}$ and write $\widetilde{Q}(\mathcal{S}) = \widetilde{Q}$ and $\Phi_{\mathcal{S}} = \Phi$. Suppose that $\mathcal{S} = Q'_1 \cup \cdots \cup Q'_{2m}$ where Q'_i are blocks of κ_{N+2} and that $\mathcal{S}$ has replaced two I-blocks $Q_1 \cup Q_2$ of κ_{N+1}.

If $m = 1$ then $Q_1 = Q'_1$, $Q_2 = Q'_2$ and Φ is the identity in $\widetilde{Q}$.

Suppose now that $m \geq 2$. We write $Q_i = \mathcal{T}^{N+2}(\ell_i, \frac{M-2}{M^2})$ and $Q'_j = \mathcal{T}^{N+2}(\ell'_j, \frac{M-2}{M^2})$ for $i = 1, 2$ and $j = 1, \dots, 2m$ where ℓ_1, ℓ_2 are I-segments, ℓ'_i is an L-segment when $i = 1, m, m+1, 2m$ and an I-segment otherwise. Let $\widehat{\ell}$ be an I-segment of length $\frac{1}{10M}$ intersecting both ℓ_1 and ℓ_2. Define $\widehat{\ell}_1 = \ell_1 \setminus \widehat{\ell}$, $\widehat{\ell}_{2m} = \ell_2 \setminus \widehat{\ell}$ and $\{\widehat{\ell}_j\}_{j=1}^{2m-1}$ be a partition of $\widehat{\ell}$ into I-segments of length $\frac{1}{(2m-2)10M}$.

Let $\Phi : \partial(\widetilde{Q} \setminus \mathcal{S}) \times I \to \widetilde{Q}$ be a PL bi-Lipschitz isotopy on $\partial(\widetilde{Q} \setminus \mathcal{S})$ such that $\Phi(\cdot, t)|\partial\widetilde{Q} = \mathrm{id}$ for all $t \in [0,1]$, $\Phi(\cdot, 0) = \mathrm{id}$, and $\Phi(\cdot, 1)|\partial\mathcal{S}$ maps each Q'_j onto $\widehat{Q}_j = \mathcal{T}(\widehat{\ell}_j, \frac{M-2}{M^2})$, while each $\Gamma_{N+2,u} \cap Q'_j$ is mapped onto $\Gamma_{N+1,u} \cap \widehat{Q}_j$ by arc-length parametrization. Let $\Sigma_t = \Phi(\partial(\widetilde{Q} \setminus \mathcal{S}), t)$.

We use the following theorem of Väisälä on bi-Lipschitz extensions.

Theorem 6.1 ([20, Corollary 5.20]). *Let $n \geq 2$ and $\Sigma \subset \mathbb{R}^n$ be a compact piecewise linear manifold of dimension n or $n-1$ with or without boundary. Then, there exist $L', L > 1$ depending on Σ, such that every L-bi-Lipschitz embedding $f : \Sigma \to \mathbb{R}^n$ extends to an L'-bi-Lipschitz map $F : \mathbb{R}^n \to \mathbb{R}^n$.*

By Theorem 6.1, for each $t \in [0,1]$, there are constants $L_t, L'_t > 1$ such that any L_t-bi-Lipschitz map $f : \Sigma_t \to \mathbb{R}^{N+2}$ has an L'_t-bi-Lipschitz extension $F : \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$. For all $t \in [0,1]$, there is an open interval Δ_t such that $1 + C|s-t| < L_t$ for all $s \in \Delta_t$. Cover $[0,1]$ with finitely many intervals $\{\Delta_{t_j}\}_{j=1}^{l}$, where $0 = t_0 < t_1 < \cdots < t_l = 1$ and $\Delta_{t_{j-1}} \cap \Delta_{t_j} \neq \emptyset$. For each $j = 1, \dots, l$ set $a_{2j} = t_j$ and $a_{2j-1} \in \Delta_{t_{j-1}} \cap \Delta_{t_j}$. Then, each $g_{a_j a_{j+1}}$ extends to a bi-Lipschitz map $G_{a_j a_{j+1}} : \mathbb{R}^{N+2} \to \mathbb{R}^{N+2}$. Hence, $G_{a_{2l-1}a_{2l}} \circ \cdots \circ G_{a_0 a_1}$ is a bi-Lipschitz self-map of $\widetilde{Q}$. The respective bi-Lipschitz maps on each $\widetilde{Q}$ can be pasted together and the resulting map is still bi-Lipschitz.

Similarly we use a bi-Lipschitz map in Q that maps κ_{N+1} onto κ_N satisfying all the properties of the previous bi-Lipschitz map. Inductively, we obtain a bi-Lipschitz map

$$\Theta' : (Q, \kappa_{N+2}) \to (Q, \kappa_1)$$

such that Θ' is identity on ∂Q, maps each $\Gamma_{N+2,u}$ on $\Gamma_{1,u}$ and every block $Q_{N+2,m}$ in the core κ_{N+2} is mapped to a block $\mathcal{T}^{N+2}(\ell, \frac{M-2}{M^2})$ where $\ell = \ell(m)$ is a straight line segment lying on J_1. Note that J_1 and all fibers $\Gamma_{1,u}$ of κ_1 are straight line segments isometric to each other.

Step 2: We straighten the images of $\Gamma_{N+2,u}$. Consider the line segments

$$\Gamma'_{N+2,u}(m) = \Theta'(\Gamma_{N+2,u} \cap Q_{N+2,m})$$

and let $\Gamma'_{N+2,u} = \bigcup_{m=1}^{M_{N+2}} \Gamma'_{N+2,u}(m)$. The family $\{\Gamma'_{N+2,u}\}_{u \in \mathfrak{C}^{N+1}}$ is a fibration of $\kappa_1 = \kappa_0(Q)$ and if u is a vertex of $\mathfrak{C}^{N+1}$ then $\Gamma'_{N+2,u}$ is an edge of κ_1. Let $\Theta'' : Q \to Q$ be a bi-Lipschitz mapping which is identity on ∂Q and linear on each $\Gamma'_{N+2,u}(m)$. Moreover, for all u, m, $\Theta''(\Gamma'_{N+2,u}(m))$ lies on $\Gamma'_{N+2,u}$ and its length is $1/M$. Define now $\Theta_z^{\mathcal{F}_0} = (\Theta')^{-1} \circ (\Theta'')^{-1} \circ \Theta_0^{\mathcal{F}_0}$ and the proof is complete.

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